Single-valuedness of wavefunctions from global gauge invariance in two-dimensional quantum mechanics



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COMMENT

# Single-valuedness of wavefunctions from global gauge invariance in two-dimensional quantum mechanics

M Bawin† and A Burnel

Institut de Physique au Sart Tilman, Université de Liège, Bâtiment B5, B-4000, Belgium

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**Abstract.** We consider in two-dimensional non-relativistic quantum mechanics an arbitrary static magnetic field confined within an inaccessible region $D$ of arbitrary geometry. We show that global gauge invariance implies that the wavefunction $\psi(x, y)$ for a charged particle outside $D$ is single-valued in a single-valued gauge.

Although Aharonov and Bohm's classic paper (Aharonov and Bohm 1959) is now over a quarter of a century old, both the theoretical interpretation of the Aharonov-Bohm (AB) effect and its experimental verification remain a controversial topic. This controversy was discussed by Peshkin (1981), Ruijsenaars (1981) and Klein (1980) in recent review papers. In particular, Henneberger (1981) proposed to describe the wavefunction of a charged particle in the field of an infinitely long and inaccessible solenoid by means of a *multivalued* wavefunction. In this scheme, the AB effect is a consequence of the multivalued character of the wavefunction, and is not interpreted as a scattering event as in the original—and subsequent—AB papers (Aharonov *et al* 1984) which use a *single-valued* description of the problem. Furthermore, the description of two-dimensional systems in terms of multivalued wavefunctions has also been advocated by Wilczek (1982a, b), with far reaching consequences: particles with multivalued wavefunctions may then carry fractional spin and obey fractional statistics. The latter point has been disputed by Lipkin and Peshkin (1982) and Jackiw and Redlich (1983) and further studied by Wu (1984), while Liang (1984) has studied multivalued wavefunctions by means of Feynman's path-integral method. On the other hand, Yang (1984) has emphasised that our current microscopic understanding of flux quantisation in superconducting rings (Byers and Yang 1961) rests upon the use of *single-valued* wavefunctions for the description of electrons inside the ring. In the analysis of Byers and Yang (1961), the electron energy levels—and hence the partition function of the system—is flux dependent. This flux dependence is crucial in order to account for flux quantisation. In contrast, in the multiple-valued wavefunction approach, electron energy levels in the ring are the same as for free electrons (Liang 1984), so that no microscopic basis for flux quantisation emerges. This is not surprising as Byers and Yang (1961) already noted the close connection between the AB effect as a scattering event and flux quantisation in superconducting rings.

In view of the above mentioned controversy and difficulties, we think it is of definite interest to give a general proof that global gauge invariance (Wu and Yang 1975,

† Chercheur qualifié FNRS.

  

Daniel and Viallet 1980) when applied to two-dimensional problems where a given inaccessible region of arbitrary geometry contains a non-zero arbitrary static magnetic field implies single-valuedness of the wavefunction in a single-valued gauge. Thus global gauge invariance agrees with the analysis developed by Aharonov and Bohm (1959) and Byers and Yang (1961), Lipkin and Peshkin (1982) and Jackiw and Redlich (1983), while it is in disagreement with the multivalued wavefunction approach proposed by Henneberger (1981) and Wilczek (1982a, b). Our proof is a generalisation of a previous result (Bawin and Burnel 1983) obtained for a cylindrical geometry. Let us emphasise that, in this particular problem, the required gauge invariance property of our system is called global by analogy with the magnetic monopole problem (Wu and Yang 1975), as we use two overlapping potentials to describe the wavefunction everywhere. For more mathematical details about global gauge invariance, see Wu and Yang (1975), Daniel and Viallet (1980) and Bawin and Burnel (1983).

In order to see this in detail, consider, as depicted in figure 1, a simply connected domain $D(x, y)$ of the $(x, y)$ plane bounded by some curve $\Gamma$. We assume that there exists a magnetic field $B(x, y)$ in the $z$ direction, extending to infinity (so that we have a purely two-dimensional system), such that $B(x, y) \equiv 0$ for $x, y \notin D(x, y)$. Consider now the single-valued gauge $A$ defined by

$$A_x = 0, \tag{1}$$

$$A_y = \int_{x_0}^{x} B(x', y)\,\mathrm{d}x', \tag{2}$$

where as shown in figure 1 the point $(x_0, y_0)$ is below and to the left of $D(x, y)$ but otherwise arbitrary. Similarly, we can introduce another gauge $A'$ defined by

$$A'_x = -\int_{y_0}^{y} B(x, y')\,\mathrm{d}y' \tag{3}$$

$$A'_y = 0. \tag{4}$$

Both $A$ and $A'$ are single-valued and are connected by the single-valued gauge transformation

$$A' = A - \nabla\Lambda, \tag{5}$$

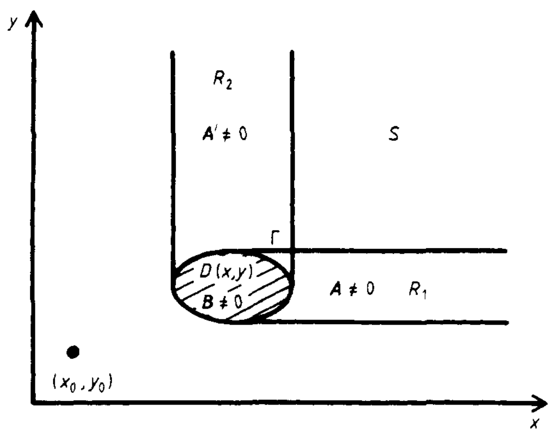


**Figure 1.** Two-dimensional problem with $B(x, y) \neq 0$ in $D(x, y)$. $D$ is inaccessible to the particle.

where

$$\Lambda(x, y) = \int_{x_0}^{x} \mathrm{d}x' \int_{y_0}^{y} \mathrm{d}y' \, B(x', y'). \tag{6}$$

We took $\Lambda$ $(x = x_0, y = y_0) = 0$ in formula (6). This merely amounts to fixing an arbitrary constant phase in the wavefunction. From (2) and (3), one finds that $\boldsymbol{A}$ is non-zero only in a horizontal strip (denoted by $R_1$ in figure 1) to the right of $D$, while $\boldsymbol{A}'$ is non-zero in a vertical strip ($R_2$) above $D$. We now wish to show that global gauge invariance implies that a solution $\psi$ to the Schrödinger equation in gauge $\boldsymbol{A}$ is continuous *everywhere* in $\complement D(x, y)$. Single-valuedness of $\psi$ is then a straightforward consequence of $\psi$ being continuous everywhere.

Let us first consider $\psi(x, y)$ for $x, y \in \complement R_1$. Since $\boldsymbol{A} = 0$, $\psi(x, y)$ is a solution to the *free* Schrödinger equation in $\complement R_1$, a simply connected domain, and must necessarily be continuous there. Consider then the wavefunction $\psi'(x, y)$ in gauge $\boldsymbol{A}'$ for $x, y \in \complement R_2$. Again, $\psi'(x, y)$ is a continuous solution to the free Schrödinger equation in $\complement R_2$. However, from global gauge invariance (5), (6), we have for any $(x, y) \in \complement D$

$$\psi'(x, y) = \exp(-\mathrm{i}e\Lambda(x, y))\psi(x, y), \tag{7}$$

where $\Lambda(x, y)$ is a continuous function.

It then follows from (7) that $\psi(x, y)$ is also continuous for $(x, y) \in \complement R_2$. Thus $\psi(x, y)$ is continuous in $\complement R_1 \cup \complement R_2$, i.e. in $\complement D$, and hence single-valued. Formula (7), which implements the notion of global gauge invariance, is obviously crucial for the argument. Since any single-valued gauge can be generated from $\boldsymbol{A}$ (or $\boldsymbol{A}'$) by a single-valued gauge transformation, we conclude that $\psi$ must be single-valued in a single-valued gauge. For completeness sake, let us note that there may be (as in figure 1) a part of strip $R_1$ which also belongs to strip $R_2$. In order to describe this region with a vanishing potential, it is sufficient to take $(x_0, y_0)$ to be now in region $S$, and define new strips below (vertical strip) and to the left (horizontal strip) of $D$. All formulae remain the same, *mutatis mutandis.*

Thus, for quite arbitrary geometry, global gauge invariance unequivocally requires $\psi$ to be single-valued in a single-valued gauge. We see no reason why this concept, while being quite relevant to more complex theories like the magnetic monopole problem (Wu and Yang 1975) or Yang–Mills theories (see Daniel and Viallet 1980), should not be relevant to a simpler theory like two-dimensional non-relativistic quantum mechanics. Finally, let us note that Silverman (1983) has proposed an experimental test of the single-valuedness of $\psi$. This experiment would then also provide a test of global gauge invariance in this particular problem.

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